A Little Mathemagic

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About the author

Rob holds a Master's degree in computer science from the University of Iowa and will someday (promise, Dad) finally get around to finishing his Ph.D. He can be reached at rob@hansen.engineering.

Why this book?

From time to time I give detailed explanations of interesting math problems to teenagers. I'm now compiling all my notes into one volume, in the hopes others might find them interesting.

How to use this book

Each chapter covers a single math problem that I think is interesting, and covers it from beginning all the way to a solution. Some chapters will use formal proofs, others will rely on casual reasoning. Each chapter stands alone and can be enjoyed independent of the others. If a chapter doesn't interest you, try another.

Each chapter includes a couple of homework questions for the benefit of teachers and parents who want to use this in an educational setting. The Appendix contains answers to each question. **Not all questions are appropriate for all readers.** Some are much harder than others. Some expect an answer, and some are meant just to get readers thinking.

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Although reasonable care has been taken to ensure accuracy, the author is not responsible for any errors.

Chapter 1

Newton's Gravity

1.1 Introduction

Imagine you wanted to send a baseball all the way to the moon. How far would you have to hit it?

The moon is about 384,000 kilometers away, but you wouldn't need to hit it that far. Somewhere between the earth and the moon, there's a *cancellation point* where the earth's gravity and the moon's gravity cancel out. If you hit the ball even one meter short of this cancellation point it will fall back to earth — but if you hit one meter beyond, the moon's gravity will draw you the rest of the way there.

So where is the cancellation point? How far would we have to hit a baseball so the moon's gravity could take it the rest of the way? With a little algebra and Newton's theory of gravity, we can figure it out.

1.1.1 The good news

Don't be scared by this chapter's size! Yes, it's very large, but that's only because each and every step is fully documented. Many steps which are glossed over in other treatments are presented here fully. We'll conquer this problem by taking a large number of very tiny steps in the direction of an answer, and each step will be fully explained.

1.1.2 Intended audience

Anyone interested in mathematics, physics, astronomy, or space travel.

1.1.3 Required skills

In order to make sense of this you'll need to have a good handle on basic algebra using one variable, square roots, and exponents. This won't teach you any algebra, but it will show you how you can use algebra to answer interesting questions about the universe.

1.2 Universal Gravitation

In 1687, Isaac Newton published a groundbreaking science book called *Philosophiæ Naturalis Principia Mathematica.* Like all science books of his day, he wrote it in Latin. Its name means, "The Mathematical Principles of Physics". (Back then, physics was called "natural philosophy".) Most people just call his book "the *Principia*".

In it, Newton put forth the following idea about how gravity works. "The force of gravity between two objects is equal to what you get by multiplying a certain constant with the masses of the two objects, and dividing that by the square of how far apart they are."

By long-standing custom, constants (things that don't change in an equation) receive capital letters. Variables (things which are allowed to change) receive lower-case letters. For right now everything is going to be fixed, so you'll see a lot of capital letters. Rewriting Newton's idea in mathematical-ese, we get:

$$F = G\frac{M_1 M_2}{D^2}$$

There are other ways this could be written. You may sometimes see this as the equivalent,

$$F = \frac{G M_1 M_2}{D^2}$$

They mean the same thing: "take the gravitational constant, multiply it by the first mass, multiply that by the second mass, and divide everything by the square of the distance between them."

1.2.1 The gravitational constant

It took many scientists many years to figure out the value of G in this equation. Today our best estimate is that $G = 6.67408 \times 10^{-11}$. That number is such a mouthful to say that most physicists will just call it G and let that letter stand in for it all. (This is sort of like letting "the *Principia*" stand in for the actual name of Newton's book, just in math-ese.)

1.2.2 An example

Imagine you have a brick and you want to know how much it weighs. You'll need to know the mass of the earth and the mass of the brick, as well as how far you are from the earth's center. The earth's mass (M_1) is 5.972×10^{24} kilograms, and standing on the earth you're about 6.371×10^{6} meters from its center (D). The brick's mass (M_2) is 1 kilogram.

How will we proceed?

Step	Formula	Explanation
1	$\overbrace{6.67408 \times 10^{-11}}^{G} \times \frac{\overbrace{5.972 \times 10^{24}}^{M_1} \times \overbrace{1}^{M_2}}{\underbrace{(6.371 \times 10^{6})^2}_{D^2}}$	Start by writing out the original equation. You can use curly brackets to make notes to remind you of what means what, if that helps you.
2	$\overbrace{6.67408 \times 10^{-11}}^{G} \times \frac{\overbrace{5.972 \times 10^{24}}^{M_1}}{\underbrace{(6.371 \times 10^{6})^2}_{D^2}}$	Any time we multiply by 1, just remove it. Multiplying by 1 never changes anything.
3	$G \times \frac{M_1}{D^2} = \frac{GM_1}{D^2}$	Any fraction $\frac{B}{C}$ can be thought of as $B \div C$. So, $A \times \frac{B}{C}$ is the same as $A \times B \div C$. Thanks to the Associative Property of Multiplication, this is the same as $(A \times B) \div C$, or $\frac{A \times B}{C}$, or $\frac{AB}{C}$.
4	$\frac{\overbrace{6.67408 \times 10^{-11}}^{G} \times \overbrace{5.972 \times 10^{24}}^{M_1}}{\underbrace{(6.371 \times 10^{6})^2}_{D^2}}$	Rewrite $\frac{GM_1}{D^2}$ using our actual numbers
5	$\frac{6.67408 \times 5.972 \times 10^{-11} \times 10^{24}}{(6.371 \times 10^{6})^2}$	$A \times B \times C \times D$ can be rearranged into $A \times C \times B \times D$ thanks to the Commutative Property of Multiplication. So long as we don't remove any numbers from our string of multiplicands, we can rearrange them how we like. In just a little bit this will make life a lot easier!
6	$\frac{6.67408 \times 5.972 \times 10^{13}}{(6.371 \times 10^{6})^2}$	Two exponential numbers with the same base can be easily multiplied together just by adding their exponents. 10^{-11} and 10^{24} both have the same base (10). $-11 + 24 = 24 - 11 = 13$, so $10^{-11} \times 10^{24} = 10^{13}$. Now, isn't that a lot nicer?

Step	Formula	Explanation
7	$6.67408 \times 5.972 \approx 39.86$	We don't need all the digits here: 39.86 is plenty accurate for us. (Scientists have a special procedure for determining how many digits to use, called "determination of significant digits". We won't get into it here: for right now, just assume two digits past the decimal point is pretty good.)
8	$\frac{39.86 \times 10^{13}}{(6.371 \times 10^6)^2}$	Another two unpleasant numbers are gone, replaced with a single much nicer number
9	$39.86 = 3.986 \times 10^1$	When using scientific notation, any number larger than ten should be broken up into a number smaller than ten multiplied by a certain power-of-ten. Any number smaller than one should be changed in a similar way.
10	$\frac{3.986 \times 10^1 \times 10^{13}}{(6.371 \times 10^6)^2}$	Re-write step 8 with the outcome of step 9
11	$\frac{3.986 \times 10^{14}}{(6.371 \times 10^6)^2}$	The numerator is now complete, and less scary.
12	$\frac{3.986 \times 10^{14}}{6.371^2 \times 10^{6^2}}$	$(a \times b)^2$ is the same as $a \times b \times a \times b$. Thanks to the Commutative Property of Multiplication, this is the same as $a \times a \times b \times b$, or $a^2 \times b^2$.
13	$6.371 \times 6.371 \approx 40.59 \approx 4.059 \times 10^1$	Multiply out the first squared term and convert it into a proper scientific-notation number
14	$\frac{3.986 \times 10^{14}}{4.059 \times 10^1 \times 10^{6^2}}$	Re-write step 12 with the outcome of step 13
15	$10^{6^2} = 10^{12}$	When raising a power to a power, multiply together the two powers. E.g., $100^3 = 1000000$, or $10^{2^3} = 10^6$.

Step	Formula	Explanation
16	$\frac{3.986 \times 10^{14}}{4.059 \times 10^{1} \times 10^{12}}$	Re-write step 14 with the outcome of step 15
17	$10^{1} \times 10^{12} = 10^{13}$	Following our rule from before ("to multiply together two exponential numbers in the same base, add their exponents"), we simplify the denominator.
18	$\frac{3.986 \times 10^{14}}{4.059 \times 10^{13}}$	Re-write step 16 with the outcome of step 17
19	$\frac{3.986 \times 10^{1}}{4.059}$	Both numerator and denominator share a factor of 10^{13}. It cancels out, leaving 10^{1} on top and nothing below.
20	$\frac{3.986 \times 10}{4.059}$	$10^{1} = 10$, so simplify step 19 with that
21	$10 \times \frac{3.986}{4.059}$	Sort of like step 3, but in reverse
22	$3.986 \div 4.059 \approx 0.982$	Plain-old long division
23	$10 \times 0.982 = 9.82$	Re-write step 21 with step 22
24	9.82	**Success!** (Huge boldface caps? Oh, yes, you deserve it!)

At the end of all this math, you've discovered that at the earth's surface the earth and a 1-kilogram brick pull on each other with a force of 9.82 somethings.

Remember: kilograms measure mass (how much of something there is), *not weight.* Weight is a measure of how hard something is being pulled down. If you were to stand on the moon your mass would remain the same (none of you would be missing), but your weight would be significantly less (you'd feel lighter).

So if the kilogram isn't a unit of weight, what is?

Scientists use the kilogram as the unit of mass, and the *newton* as the unit of weight. (You'll also see the newton called the unit of force. It's the same thing, really: weight is just "the force with which gravity is pulling you down".)

So, at the earth's surface, you've shown a 1-kilogram block weighs $9.82N$. The capital N is used as the abbreviation for newton, much in the same way "lb." is used for pound, or "kg" is used for kilogram.

1.2.3 Homework questions

1. If you wanted to find out how hard gravity would pull down a 1-kg brick on the moon, what would you need to know about the moon before you could start solving the problem?

2. So how hard should a 1-kilogram brick be pulled down on the surface of the moon? (You can use Google to find the numbers you need.)

3. Repeat it for Mars. Which of the three (the earth, the moon, and Mars) has the strongest gravity at their surface? The moon's gravity is about what fraction of the earth's? The gravity of Mars is about what fraction of the earth's?

4. In our equations we're careful to use the distance from the earth's (or the moon's, or Mars's) center. But if you were to ask a person on the street how far they are from the earth, they'd say they're standing on it. What would happen if they were truly zero distance away? (Hint: Newton's theory of gravity stops working — why?)

1.3 Finding where gravity cancels out

To find where gravity cancels out we need to find where the force of earth's gravity is exactly countered by the force of the moon's gravity. Once more, let's break out our trusty 1-kilogram brick (a useful companion in almost any physics experiment!) and figure out where the forces on it are equal.

To help keep things straight ("is that the force from the earth, or from the moon?") we'll use two special symbols: e will denote the earth, o will denote whatever object we're looking to balance between the earth and the moon, and m will denote the moon. M_e would be, for instance, "the mass of the earth", and M_o would be, "the mass of the object between the earth and the moon".

We'll let D be the total distance between the center of the earth and the center of the moon. If the brick is x meters from earth, it will be $D - x$ meters from the moon.

First, we write out the formula for the force on the brick from the earth and the moon:

Step	Formula	Explanation
1	$G\frac{M_e M_o}{x^2} = G\frac{M_m M_o}{(D-x)^2}$	The force of Earth's gravity on the object x meters away from the Earth's center equals the force of the moon's gravity on the object $D - x$ meters away from the moon's center
2	$GM_o\frac{M_e}{x^2} = GM_o\frac{M_m}{(D-x)^2}$	Use the Associative Property of Multiplication to move our M_o term out front with the G term
3	$\frac{M_e}{x^2} = \frac{M_m}{(D-x)^2}$	Divide both sides of the equation by GM_o
4	$\frac{\sqrt{M_e}}{x} = \frac{\sqrt{M_m}}{D-x}$	Run a square root over both sides
5	$\frac{(D-x)\sqrt{M_e}}{x} = \sqrt{M_m}$	Multiply everything through by $D-x$
6	$\frac{D-x}{x} = \frac{\sqrt{M_m}}{\sqrt{M_e}}$	Divide everything through by $\sqrt{M_e}$
7	$\frac{D-x}{x} = \sqrt{\frac{M_m}{M_e}}$	Normalize the fraction on the right
8	$\frac{D}{x} - \frac{x}{x} = \sqrt{\frac{M_m}{M_e}}$	Separate the left-hand side into two fractions
9	$\frac{D}{x} - 1 = \sqrt{\frac{M_m}{M_e}}$	$\frac{x}{x} = 1$
10	$\frac{D}{x} = 1 + \sqrt{\frac{M_m}{M_e}}$	Add 1 to both sides

Step	Formula	Explanation
11	$D = x\left(1 + \sqrt{\frac{M_m}{M_e}}\right)$	Multiply both sides by x
12	$\frac{D}{1 + \sqrt{\frac{M_m}{M_e}}} = x$	Divide both sides by $1 + \sqrt{\frac{M_m}{M_e}}$
13	$x = \frac{D}{1 + \sqrt{\frac{M_m}{M_e}}}$	Flip sides

We've now determined that for *any* two bodies of mass, we can find a cancellation point between them by dividing the distance between them by one plus the square root of their mass ratio!

M_m is about 7.35×10^{22} kilograms; M_e is about 5.972×10^{24} kilograms; and they're separated by about 3.844×10^8 meters.

Step	Formula	Explanation
1	$\frac{3.844 \times 10^8}{1 + \sqrt{\frac{7.35 \times 10^{22}}{5.972 \times 10^{24}}}}$	Re-write our derived equation, this time with our real numbers
2	$\sqrt{\frac{7.35 \times 10^{22}}{5.972 \times 10^{24}}}$	Move that awful fraction out so we can wrestle with it by itself
3	$\sqrt{\frac{7.35}{5.972 \times 10^{2}}}$	Cancel out a common factor of 10^{22}: already it looks a lot nicer!
4	$\sqrt{\frac{7.35}{5.972} \times \frac{1}{10^2}}$	Separate it out
5	$\sqrt{\frac{7.35}{5.972}} \times \sqrt{\frac{1}{10^2}}$	Separate it into two square root terms

Step	Formula	Explanation
6	$\sqrt{\frac{7.35}{5.972}} \times \frac{1}{10}$	Simplify the second square root
7	$0.1 \times \sqrt{\frac{7.35}{5.972}}$	Reorganize, using the fact $0.1 = \frac{1}{10}$ and the Commutative Property of Multiplication
8	$7.35 \div 5.972 \approx 1.231$	long division to the rescue
9	$0.1 \times \sqrt{1.231}$	rewrite step 7 with step 8
10	0.1×1.11	I admit, I used a pocket calculator here; 1.11 is an approximate value
11	$0.1 \times 1.11 = 0.111$	complete step 10
12	$\sqrt{\frac{7.35 \times 10^{22}}{5.972 \times 10^{24}}} \approx 0.111$	Equate step 2 with the outcome of step 11
13	$\frac{3.844 \times 10^{8}}{1 + 0.111}$	Re-write step 1
14	$\frac{3.844 \times 10^{8}}{1.111}$	Simplify the denominator
15	$10^{8} \times \frac{3.844}{1.111}$	Wow. Isn't this a *much* nicer fraction?
16	$10^{8} \times 3.46$	Long division to the rescue once again
17	3.46×10^{8}	The distance to the gravity cancellation point, in meters

We've found a gravity cancellation point 3.46×10^8 meters from the earth, or about ninety percent of the way to the moon. If we can just get there, the moon's gravity will take over and draw us the rest of the way in.

1.3.1 Homework problems

1. How far is 3.46×10^8 meters, in kilometers?

2. How far is 3.46×10^8 meters, in miles? (Hint: there are about 1.61 kilometers per mile.)

3. There are five lunar libration points. These are also called "Lagrange points", after the astronomer who discovered them, Joseph-Louis Lagrange. They're named L1, L2, L3, L4, and L5. If something is positioned exactly on a Lagrange point, it will be effectively immobile relative to the earth and moon. This is very much like a gravity cancellation point, but it's not quite identical.

   Look up where the L1 point is. You'll discover that our gravity cancellation point is almost, **but not quite**, the same as L1. There must be something else going on with the earth and the moon that's affecting our numbers and throwing them off. What could it be?

4. Do some more reading on Lagrange points. How is a Lagrange point different from a gravity cancellation point?

5. Computing a Lagrange point accurately requires solving a *quintic equation*. What's a quintic equation? Is there any general method to solve one?

Answers to Homework Questions

This section is meant for teachers and parents, and for students *after* they've tried to solve the homework problems on their own.

§1.2.3

1. The moon's mass in kilograms (7.35×10^{22}), and the moon's radius in meters (1.737×10^{6}).

2. Repeating the same process we used to compute the earth's gravity, we discover a 1-kilogram brick on the moon would be pulled down with a force of 1.63N. This is almost exactly one-sixth the force felt on the earth.

3. Mars's mass is 6.42×10^{23} kilograms; its radius, 3.39×10^{6} meters. This means a 1-kilogram brick on its surface weighs 3.73N, or about two-fifths what it weighs on earth's surface.

4. When $D = 0$, the denominator in the equation is also zero. Dividing by zero is not allowed in mathematics, so whenever a physical theory divides by zero you can be certain either the theory stopped being accurate or else the universe just got very interesting. A common joke among physicists is that "black holes are where the Almighty divides by zero".

§1.3.1

1. To convert from meters to kilometers, divide by 1000. You can do this easily by seeing that $1000 = 10^3$. Then just use the subtract-the-exponents trick to do the division: $3.46 \times 10^8 \div 10^3 = 3.46 \times 10^5$ kilometers, or 346,000 kilometers.

2. $3.46 \times 10^5 \div 1.61 = 10^5 \times \frac{3.46}{1.61}$. That fraction is about 2.15, so the cancellation point is about 215,000 miles away.

3. Lagrange points aren't the same as gravity cancellation points, but they're similar. Lagrange points are misnamed: they're actually Lagrange *orbits*. Something

at one of the earth's Lagrange points is still traveling in a circle (an ellipse, really) around the earth; it's just that at each step along its orbit it preserves the exact same distances between itself, the earth, and the moon.

In our derivation we assumed the earth and the moon were fixed in space. In reality, though, the moon is in constant motion around the earth. The Lagrange points represent cancellation in a moving system. We computed cancellation points in an unmoving system — what physicists would call a *static* system.

This doesn't mean we're wrong! Quite the opposite. We correctly computed the gravity cancellation point. If you can hit a baseball (or launch a rocket) to that point, the moon's gravity **will** take it the rest of the way.

All it means is that we're computing a straight-line journey between the earth and the moon. A Lagrange point is a stable orbit around the earth, which is to say it's a circle around the earth without a beginning or ending.

Lagrange points are closely related to cancellation points, but they're also quite distinctly different.

4. See previous answer.

5. A quintic equation is one where there's a fifth-power variable. Even simple quintic equations, such as $x^5 - x + 1 = 0$, can be murderously hard to solve with algebra. One of the major motivating forces behind the invention of calculus was the need for a more powerful kind of math that could address problems like these.